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Skyrmions on S^3 and H^3 from instantons

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Abstract. A recent proposal to derive Skyrme fields in flat space from Yang–Mills instantons is modified to obtain Skyrme fields on the 3-sphere and on hyperbolic 3-space. Good approximations to the single Skyrmion on these spaces are found.

1. Introduction

To describe nucleons and their interactions in the Skyrme model it is necessary to find sensible truncations of the full field theory. In a given baryon number sector one seeks a finite-dimensional manifold of Skyrme fields whose coordinates are the physically relevant degrees of freedom at low energy. For the single Skyrmion this is straightforward, and the appropriate collective coordinates are well known. The two-Skyrmion sector presents greater difficulties. A candidate manifold for this case has been proposed in [1].

Recently it has been suggested that good approximations to such manifolds may be obtained by taking the holonomy of Yang–Mills instantons [2]. The construction is as follows. Let $A_i(x)$ be an SU(2) gauge field in $\mathbb{R}^4$, satisfying boundary conditions consistent with conformally compactifying $\mathbb{R}^4$ to S^4. One defines the associated SU(2)-valued Skyrme field in $\mathbb{R}^3$, $U(\boldsymbol{x})$, to be the holonomy of $A_i(x)$ along all lines parallel to the time axis. Formally,

$$U(\boldsymbol{x}) = \mathcal{P}\exp\left(-\int_{-\infty}^{\infty} A_\tau(\boldsymbol{x},\tau)\mathrm{d}\tau\right) \tag{1.1}$$

where $\mathcal{P}$ denotes path-ordering and we have written $x = (\boldsymbol{x},\tau)$. The boundary conditions ensure that $U(\boldsymbol{x}) \to 1$ as $|\boldsymbol{x}| \to \infty$. If A_i has charge k, then U is a Skyrme field of baryon number $B = k$. Under a gauge tranformation g_x, U transforms to $g_\infty^{-1} U g_\infty$. Thus the definition of U is gauge invariant up to the constant gauge transformations, which produce $\mathrm{SO(3)}_{\mathrm{isospin}}$ rotations of the Skyrme fields. Now let A_i be an instanton field. Allowing for the constant gauge transformations, the space of instantons of charge k is an $8k$-dimensional manifold, M_k. Since time-translating A_i leaves U unchanged, M_k generates an $(8k-1)$-dimensional manifold of Skyrme fields with baryon number $B = k$.

The 1-instantons generate a seven-dimensional manifold of $B = 1$ Skyrme hedgehogs, the coordinates being the position, orientation and scale size. In standard position the Skyrme field is

$$U(\boldsymbol{x}) = \exp(\mathrm{i}f(r)\hat{\boldsymbol{x}}\cdot\boldsymbol{\tau}) \tag{1.2}$$

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where

$$f(r)=\pi\left[1-\left(1+\frac{\lambda^2}{r^2}\right)^{-1/2}\right]. \tag{1.3}$$

The minimum value of the Skyrme energy of this field is attained when $\lambda^2 \simeq 2.11$ and is less than 1% greater than the energy of the true $B=1$ Skyrmion. The 2-instantons, and the 15-dimensional manifold of $B=2$ Skyrme fields they generate, are described in [2]. This is the case of interest for two-nucleon dynamics. A modification of the Skyrme field (1.3) is obtained from the holonomy of a periodic, or thermal, instanton, integrated along one period. The energy is very slightly reduced for the optimal parameters [3].

In this paper we consider generating Skyrme fields by taking the holonomy along different sets of curves. In any such construction, it is necessary for the curves to be the orbits of a one-parameter subgroup G of the conformal group SO(5, 1) acting on $\mathbb{R}^4$. The holonomies then give a Skyrme field on a space which, if it is a manifold at all, has a metric well-defined up to scaling. Consider the following possibilities.

(1) G is the group of time-translations. Its orbits—the time-lines—realise the equivalence $\mathbb{R}^4 \sim \mathbb{R}^3 \times \mathbb{R}$ and one obtains Skyrme fields on the quotient space $\mathbb{R}^3$. This is the flat-space construction described above.

(2) G is the group of dilations. The orbits are the rays based at the origin; they realise the conformal equivalence $\mathbb{R}^4 \setminus \{0\} \sim S^3 \times \mathbb{R}$, and one obtains Skyrme fields on the 3-sphere, S^3.

(3) G is an appropriate SO(2) group. The orbits are then the circles appearing in the conformal equivalence $\mathbb{R}^4 \setminus S^2 \sim H^3 \times S^1$, and Skyrme fields on hyperbolic 3-space result.

In the following we investigate the cases (2) and (3). We shall find that from the $k=1$ instanton we obtain good approximations to the true $B=1$ Skyrmion on S^3 and H^3. Before the calculations, two general remarks. Firstly, note that the conformal symmetry of the Yang–Mills equations means that we are free to employ any conformally equivalent description of the constructions. Hence the set of Skyrme fields generated by M_k is independent of the way the 3-manifold is realised as a quotient space. Secondly, observe that when one takes holonomies the conformal group acting on M_k is broken down to the isometry group of the 3-manifold (E_3, SO(4), SO(3, 1) in the three cases described) together with rescalings; this residual group commutes with G. The action of G on M_k leaves the Skyrme fields unchanged, i.e. the orbits of G in M_k are the sets of instantons giving rise to the same Skyrme field.

2. Skyrme fields on S^3

As stated above, we may obtain Skyrme fields on the 3-sphere by taking the holonomy along all rays based at the origin in $\mathbb{R}^4$ (figure 1(*a*)). Let (μ,θ,ϕ) be hyperspherical coordinates on S^3. The Skyrme field on S^3 generated by the gauge field $A_i(x)$ is given formally by

$$U(\mu,\theta,\phi)=\mathcal{P}\exp\left(-\int_{\gamma(n)} A_i(x)\mathrm{d}x_i\right). \tag{2.1}$$

Here $\gamma(n)$ is the path along the ray $\{x_i = sn_i : s \in [0,\infty)\}$, where n_i is the unit vector $(\sin\mu\sin\theta\cos\phi, \sin\mu\sin\theta\sin\phi, \sin\mu\cos\theta, \cos\mu)$. Explicitly, one must solve

$$\frac{\partial\tilde{U}}{\partial s}\tilde{U}^{-1} = -A_i n_i \tag{2.2}$$

along each ray, with initial data $\tilde{U}(s=0) = 1$; then $U = \tilde{U}(s=\infty)$. Note that under gauge transformations, $U \to g_0^{-1} U g_\infty$, i.e. an SO(4) group of chiral transformations survives. For a general Skyrme field on the 3-sphere these are physical zero modes as there is no analogue of the requirement for Skyrme fields in $\mathbb{R}^3$ that $U(\boldsymbol{x}) \to 1$ as $|\boldsymbol{x}| \to \infty$.

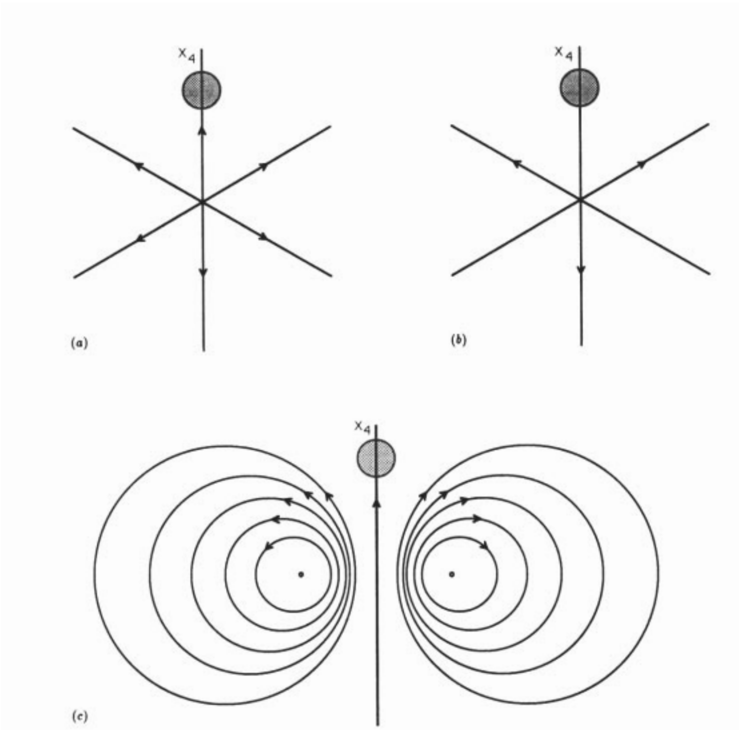

Figure 1. The curves along which the holonomies are taken, in a plane in $\mathbb{R}^4$ containing the 4-axis. The instanton is centred on the 4-axis and is shown schematically, shaded. (a) The rays based at the origin; (b) the lines through the origin; (c) the coaxal system of circles.

We consider just the $k=1$ instantons. The 't Hooft formula is [4]

$$A_i(x) = \frac{i}{2}\bar{\eta}_{aij}\tau^a\frac{\partial_j\rho}{\rho} \qquad \rho = 1 + \frac{\lambda^2}{(x_i - c_i)^2} \tag{2.3}$$

where $\bar{\eta}_{aij} (a = 1,2,3)$ is the anti-self-dual tensor $\varepsilon_{aij4} - \delta_{ai}\delta_{j4} + \delta_{aj}\delta_{i4}$, and where c_i is the centre of the instanton, and $\lambda (> 0)$ its scale. Without loss of generality, we may take $c_i = (\mathbf{0}, c)$ with $c \geq 0$. Varying c and λ, together with SO(4) rotations in $\mathbb{R}^4$, generates the eight-parameter set of $k=1$ instantons, M_1.

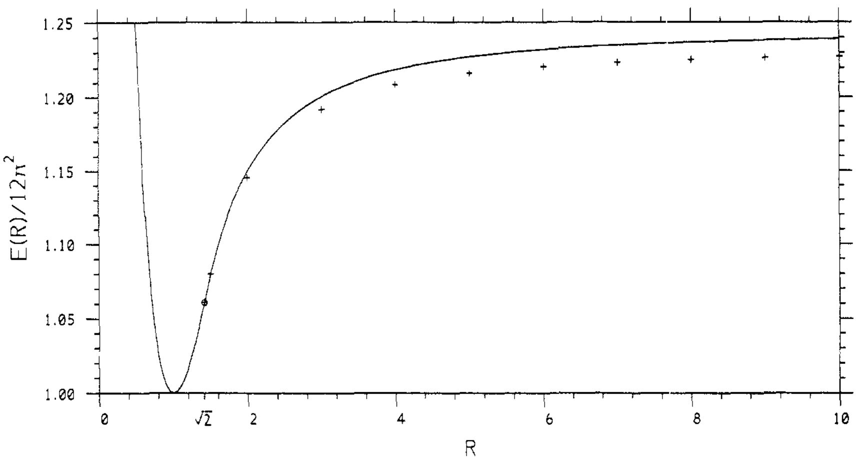

Figure 2. The Skyrmion on $S^3(R)$: The curve is the energy of the optimised profile (2.5); the +'s indicate values of the true energy (from [7]). Below the critical radius $R = \sqrt{2}$ ($\oplus$), the optimised profile is exact.

With $A_i(x)$ given by (2.3), solving (2.2) reduces to ordinary integration, just as in [2], and we obtain

$$U(\mu, \theta, \phi) = \exp\,(\mathrm{i}f(\mu)\hat{\boldsymbol{n}} \cdot \boldsymbol{\tau}) \tag{2.4}$$

where $\hat{\boldsymbol{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, and

$$f(\mu) = \pi - \mu - \frac{\sin\mu}{\sqrt{\sin^2\mu + \sigma^2}}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{\cos\mu}{\sqrt{\sin^2\mu + \sigma^2}}\right)\right] \qquad \sigma = \frac{\lambda}{c} \tag{2.5}$$

i.e. a $B = 1$ Skyrme hedgehog at the north pole, in standard orientation. The parameter σ appearing in the profile (2.5) measures the size; allowing also for the SO(4) rotations, which change the position and orientation of the Skyrme hedgehog on the 3-sphere, we have the expected seven-parameter family of Skyrme fields. The Skyrme fields are invariant under $(\lambda, c) \to (a\lambda, ac)$; this is just the action of G = {dilations} on M_1. Note that (2.5) only holds for $c \neq 0$. For the case $c = 0$ one must take the $c \to 0$ (i.e. $\sigma \to \infty$) limit; in this limit, the holonomy comes entirely from the singularity in $A_i(x)$ at the point $x_i = 0$. The resulting profile is $f(\mu) = \pi - \mu$, which corresponds to the identity map from S^3 to SU(2).

Thus far we have not specified the radius of the 3-sphere; in order to define the energy of the Skyrme fields we must now do so. The Skyrme energy of a hedgehog field on $S^3(R)$, the 3-sphere of radius R, is

$$E = 4\pi\int_0^\pi \left\{R\sin^2\mu\left[\left(\frac{\mathrm{d}f}{\mathrm{d}\mu}\right)^2 + 2\frac{\sin^2 f}{\sin^2\mu}\right] + \frac{1}{R}\sin^2 f\left[\frac{\sin^2 f}{\sin^2\mu} + 2\left(\frac{\mathrm{d}f}{\mathrm{d}\mu}\right)^2\right]\right\}\mathrm{d}\mu. \tag{2.6}$$

For a given R, one may calculate the energy for the profile (2.5) as a function of σ. We have computed the minimum energy $E(R)$ (and the corresponding scale $\sigma(R)$) numerically over a range of R. Some of the values are shown in figure 2. We find that for R less than the critical radius $\sqrt{2}$, $E(R)$ is precisely the energy of the true Skyrmion, namely $6\pi^2(R + R^{-1})$. For R greater than this, it is too large, with the error increasing monotonically to that of the flat-space profile as $R \to \infty$. We clarify these results with a few remarks.

Regarding the Skyrme field as a map $S^3(R) \to S^3(1)$, recall that for $R \leq \sqrt{2}$ the true minimal-energy Skyrmion is, up to SO(4) rotation, the identity map; and that for $R = \sqrt{2} + \varepsilon\,(\varepsilon \ll 1)$, it is a conformal map to $O(\varepsilon^{1/2})$ [5,6]. Now, consider the $\sigma \gg 1$ limit of (2.5):

$$f(\mu) \sim \pi - \mu - \frac{\pi}{2\sigma}\sin\mu. \tag{2.7}$$

To $O(1/\sigma)$, this also gives a conformal map, reducing to the identity map (up to an SO(4) rotation) for $\sigma = \infty$. As a consequence, after minimising the energy, the profile (2.5) is exact for $R \leq \sqrt{2}$ and correct to $O(\varepsilon^{1/2})$ for $R = \sqrt{2} + \varepsilon$. The energy to $O(\varepsilon^2)$ is

$$E(R) = 12\pi^2\left(\frac{3}{2\sqrt{2}} + \frac{\varepsilon}{4} - 0.261\varepsilon^2\right). \tag{2.8}$$

The first two terms agree with the conformal map and are exact. The $O(\varepsilon^2)$ term—a fit with the numerical results—is rather better than that of the conformal map, which is $-\varepsilon^2/4\sqrt{2} = -0.177\varepsilon^2$.

Turning now to the limit $R \to \infty$, set $\boldsymbol{x} = R\mu\,\hat{\boldsymbol{n}}$ and let $\mu \to 0$ so that $r = |\boldsymbol{x}|$ remains finite. From (2.4) and (2.5), keeping $R\sigma$ finite, we obtain

$$U(\boldsymbol{x}) = \exp\,(\mathrm{i}\,f(r)\hat{\boldsymbol{x}}.\boldsymbol{\tau}) \qquad \text{where} \quad f(r) \sim \pi\left[1 - \left(1 + \frac{R^2\sigma^2}{r^2}\right)^{-1/2}\right] \tag{2.9}$$

in agreement with the flat-space profile (1.3). Asymptotically, we find

$$E(R) \sim 12\pi^2\left(1.243\,198 - \frac{0.431}{R^2} + \frac{0.26}{R^3}\right). \tag{2.10}$$

The first term is the energy of the flat-space profile; the second may be obtained in terms of definite integrals of the flat-space profile and its derivatives. The last term is a fit with the numerical results. The corresponding formula for the true Skyrmion is [7]

$$E_{\mathrm{Sk}}(R) \sim 12\pi^2\left(1.231\,445 - \frac{0.419}{R^2} + \frac{0.25}{R^3}\right). \tag{2.11}$$

We conclude this section by noting an interesting variation of the construction. This is to take the holonomy along not each ray, but each full line $\{x_i = \tau n_i : \tau \in (-\infty, \infty)\}$ (figure 1(*b*)). Expression (2.1) now defines a Skyrme field on S^3 with $B = 0$, independently of the charge k of $A_i(x)$. For the 1-instanton (2.3), the profile function becomes

$$f(\mu) = \pi\left[1 - \left(1 + \frac{\sigma^2}{\sin^2\mu}\right)^{-1/2}\right] \tag{2.12}$$

with $\sigma = \lambda/c$ as before. If σ is small this clearly describes a Skyrmion at the north pole of S^3 and an anti-Skyrmion at the south pole. Expression (2.12) shows that the magnitude of the holonomy along a full line depends on a single geometric quantity:

the scale of the instanton, λ, divided by the distance of the line from the instanton, $c\sin\mu$. In the flat-space construction the distance is just r; replacing $c\sin\mu$ by r, we recover the flat-space profile (1.3).

As with (2.5), we would like to see how well (2.12) works as an ansatz for true solutions of the Skyrme model. Accordingly, we have investigated numerically the $B = 0$ solutions of the hedgehog form (2.4), and with reflection symmetry $f(\pi - \mu) = f(\mu)$, over a range of R (see figure 3). Even this restricted sector has a rich structure, with infinitely many new solutions of increasing energy emerging as the 3-sphere grows larger. When R is small, the only solution is the zero-energy solution $f(\mu) = 0$. The first pair of new solutions appears at a cusp bifurcation at $R \simeq 2.77$. The lower-energy solution is stable within the hedgehog sector (though unstable to general perturbations) and describes a Skyrmion and an anti-Skyrmion localised at antipodal points of the 3-sphere. Its energy is twice that of a single Skyrmion on $S^3(R)$, less a binding energy which tends to zero as $R \to \infty$. The higher-energy solution is unstable even within the hedgehog sector. Its energy is spread over the whole 3-sphere even for large R. As $R \to \infty$, its profile $f(\mu)$ tends to a fixed form and the energy increases linearly with R.

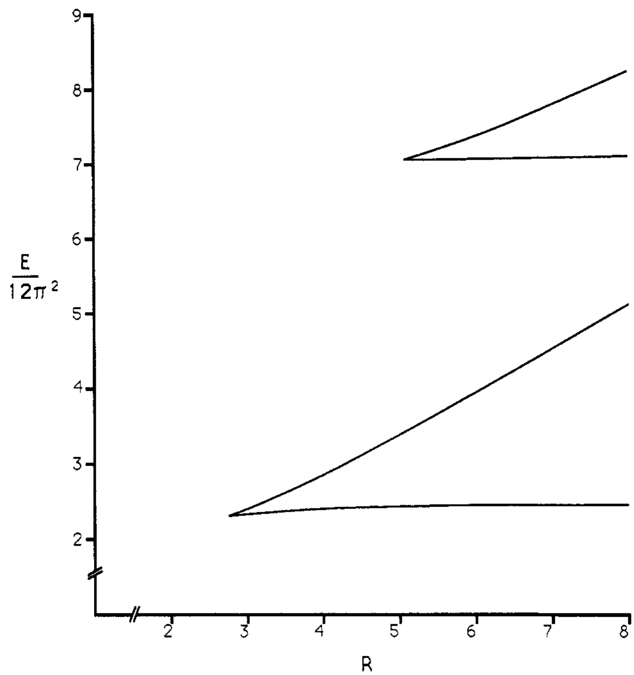

Figure 3. The spectrum of $B = 0$ solutions on $S^3(R)$ of the hedgehog form (2.4), with reflection symmetry $f(\pi - \mu) = f(\mu)$.

Now let us compare the extrema of the energy for a profile of the form (2.12). For $R \lesssim 2.82$, the energy increases monotonically with σ and there is only the trivial solution $f(\mu) = 0$. For $R \gtrsim 2.82$ there are two new extrema, a local maximum and minimum. These correspond to the true solutions described above. The energies at these extrema, together with those of the true solutions, are shown in table 1 for various values of R. Once again it is found that the instanton-derived ansatz performs rather well. To obtain reasonable approximations to the solutions that emerge at the second cusp bifurcation at $R \simeq 5.08$, and at subsequent bifurcations, one probably needs to use ansätze derived from instantons of charge greater than 1.

Table 1. The energy of the two $B = 0$ solutions on $S^3(R)$ described in the text, and the corresponding results for the optimised ansatz (2.12).

R	$E_{\text{true}}/12\pi^2$	$E_{\text{ansatz}}/12\pi^2$
3	2.346	2.359
	2.388	2.389
4	2.405	2.425
	2.846	2.852
5	2.427	2.449
	3.380	3.394
6	2.439	2.461
	3.944	3.967
7	2.445	2.468
	4.524	4.556
8	2.449	2.473
	5.113	5.153
∞	2.463	2.486
	$\sim 0.616R$	$\sim 0.623R$

3. Skyrme fields on H^3

To obtain Skyme fields on H^3 we take the holonomy along a set of circles generated by an appropriate SO(2) action on $\mathbb{R}^4$. A careful choice of coordinates renders the holonomy of the field (2.3) essentially Abelian, and so computable by ordinary integration, as in the previous cases.

It is convenient to work in toroidal coordinates (μ, θ, ϕ, ν) in $\mathbb{R}^4$. These are defined by

$$x_i = \frac{1}{\cosh\mu + \cos\nu}(\sinh\mu\sin\theta\cos\phi, \sinh\mu\sin\theta\sin\phi, \sinh\mu\cos\theta, \sin\nu) \ . \tag{3.1}$$

Here θ and ϕ are the coordinates of a 2-sphere; $\mu \in [0, \infty)$ and $\nu \in [0, 2\pi)$. Our circles are the lines of constant (μ, θ, ϕ), parametrised by ν. Note that in any plane containing the 4-axis the circles form a coaxal system (i.e. a linear family) of non-intersecting type with linear parameter $1/\tanh\mu$ [8]. (See figure 1(c) and equation (4.1) below.) The $\mathbb{R}^4$ metric is

$$ds^2 = \frac{1}{(\cosh\mu + \cos\nu)^2}\left\{\left[d\mu^2 + \sinh^2\mu(d\theta^2 + \sin^2\theta d\phi^2) +\right] d\nu^2\right\} . \tag{3.2}$$

The 3-metric $\{\cdot\}$ is just the metric of H^3, in spherical polar coordinates (μ, θ, ϕ). Thus (3.2) is a realisation of the conformal equivalence $\mathbb{R}^4 \setminus S^2 \sim H^3 \times S^1$, and the holonomies along the circles give a Skyrme field on H^3.

To find the holonomies we solve

$$\frac{\partial \tilde{U}}{\partial \nu}\tilde{U}^{-1} = -A_i \frac{\partial x_i}{\partial \nu} \tag{3.3}$$

for $U = \tilde{U}(\nu = 2\pi)$, taking $\tilde{U}(\nu = 0) = 1$. Once again we should consider the effect of gauge transformations. U now undergoes *local* isospin rotations:

$$U(\mu, \theta, \phi) \to g_p^{-1} U(\mu, \theta, \phi) g_p \tag{3.4}$$

where p denotes the point $(\mu, \theta, \phi, \nu = 0)$ in $\mathbb{R}^4$. We may restore the gauge invariance of the construction by stipulating that within any class of gauge related Skyrme fields we are to consider only those of minimal energy. Then only an SO(3) of rigid isospin rotations survives, just as in flat space.

Solving (3.3) for an instanton centred on the 4-axis, with A_i given by (2.3), we obtain the hedgehog (2.4) (where now of course μ, θ, ϕ are coordinates on H^3) with profile

$$f(\mu) = \pi \left[1 - \left(1 + \frac{\sigma^2}{\sinh^2 \mu}\right)^{-1/2}\right] \tag{3.5}$$

where

$$\sigma = \frac{2}{\lambda + (1 + c^2)\lambda^{-1}}. \tag{3.6}$$

Note that σ is restricted to lie in the interval $(0, 1]$, and that the full range is only accessible when $c = 0$. The full set of 1-instantons may be generated from those on the 4-axis by the action of an SO(3, 1) subgroup of the conformal group. The corresponding action on the holonomies changes the position and orientation of the hedgehog in H^3. (This is just the SO(3, 1) that commutes with G = SO(2).) Once again we obtain a seven-parameter family of Skyrme fields.

When $c = 0$ and $\lambda = 1$, and hence $\sigma = 1$, the instanton is invariant under the $G =$ SO(2) group we have chosen. The same applies to any instanton obtainable from this one by the action of the SO(3, 1). This is most easily understood in a conformally equivalent description on the 4-sphere. After a suitable stereographic projection, the instanton is uniform over the 4-sphere and hence SO(5) rotationally invariant, and G is an SO(2) subgroup of this SO(5) whose orbits on the 4-sphere are all the circles parallel to one great circle. An SO(2)-invariant instanton may be identified with a charge-1 monopole in hyperbolic space [9]. The component of the gauge potential $A_\nu = A_i \partial x_i / \partial \nu$ may be identified with the monopole's Higgs field, and is independent of ν. When $\sigma = 1$, the Skyrme field profile (3.5) reduces to

$$f(\mu) = \pi(1 - \tanh \mu) \tag{3.7}$$

which is consistent with what one obtains by exponentiating the Higgs field of a hyperbolic monopole.

The Skyrme energy functional on hyperbolic space of curvature $K = -R^{-2}$ is given by the expression (2.6), but with $\sin \mu$ now replaced by $\sinh \mu$, and the integration range running from 0 to ∞. We have computed the energy of the true $B = 1$ Skyrmion on $H^3(R)$, and also that of the optimised ansatz (3.5), over a range of R. The optimal value of σ increases with $|K|$, and the upper limit, $\sigma = 1$, is saturated at $|K| \simeq 2.8$. It is therefore natural to allow σ to vary freely, even though the values $\sigma > 1$ no longer arise from SU(2) instantons. The results are shown in figure 4. If one makes the restriction $\sigma \leq 1$, then the energy of the ansatz rises rather more steeply when $|K| > 2.8$, so does not perform as well. The asymptotic formulae for small curvature (large R) corresponding to (2.10) and (2.11) are

$$E(R) \sim 12\pi^2 \left(1.243\,198 + \frac{0.431}{R^2}\right) \tag{3.8}$$

for the ansatz, and

$$E_{\mathsf{Sk}}(R) \sim 12\pi^2 \left(1.231\,445 + \frac{0.419}{R^2}\right) \tag{3.9}$$

for the true Skyrmion. As before, the $1/R^2$ terms can be calculated exactly—they are the same as in (2.10)–(2.11) but with opposite sign—but our numerical results are not sufficiently accurate to fit the next-order term reliably.

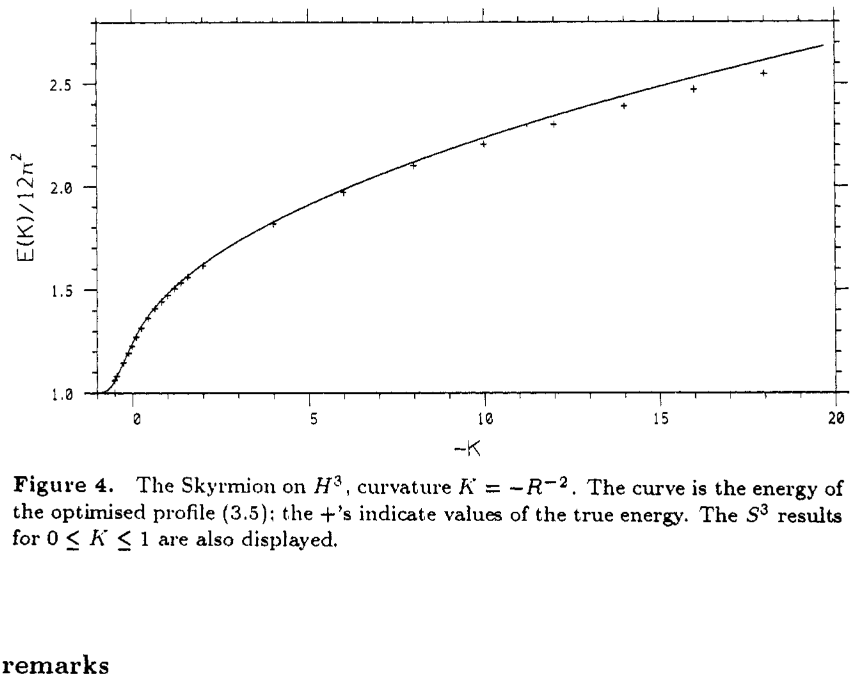

Figure 4. The Skyrmion on H^3, curvature $K = -R^{-2}$. The curve is the energy of the optimised profile (3.5); the +'s indicate values of the true energy. The S^3 results for $0 \leq K \leq 1$ are also displayed.

4. Further remarks

We conclude with some remarks on the profile (3.5) and its extension to the values $\sigma > 1$. We can interpret $\sigma/\sinh\mu$ in a geometrical way, just as we did for the corresponding quantities in the similar profiles (1.3) and (2.12). Expression (3.5) is the magnitude of the holonomy along the circle

$$\left(r - \frac{1}{\tanh\mu}\right)^2 + x_4^2 = \frac{1}{\sinh^2\mu} \qquad \theta, \phi = \text{constant} \tag{4.1}$$

where $r = |\boldsymbol{x}|$. It is straightforward to conformally transform any given circle (4.1) into a straight line. The instanton will suffer a transformation too. One finds that the scale of the (transformed) instanton divided by its distance from the line yields $\sigma/\sinh\mu$ with σ given by (3.6). Thus one has the same interpretation as before.

It is also interesting to try to find a conformally equivalent description in which *all* the circles become straight lines. To this end, note that all the circles, though non-intersecting in real space, pass through the pair of complex points $(\mathbf{0}, \pm\mathrm{i})$. The required transformation is essentially an inversion in a 3-sphere centred at one these points, $(\mathbf{0}, \mathrm{i})$ say. Let us extend the coordinates x_i to complex values, so that we are now working in the space $\mathbb{C}^4$, and employ the conformal transformation

$$
\begin{aligned}
x_j &\to \frac{2x_j}{r^2 + (x_4 - \mathrm{i})^2} \qquad j = 1, 2, 3 \\
x_4 &\to \frac{-2(x_4 - \mathrm{i})}{r^2 + (x_4 - \mathrm{i})^2} + \mathrm{i}.
\end{aligned}
\tag{4.2}
$$

The circles (4.1) become lines:

$$
x_4 = \frac{\mathrm{i}}{\tanh\mu} r \qquad \theta, \phi = \text{constant}. \tag{4.3}
$$

Now consider the instanton (2.3), which we take to be at the origin ($c = 0$). After the transformation, it has scale $2\lambda/(\lambda^2 - 1)$ and centre $(\mathbf{0}, \mathrm{i}(\lambda^2 + 1)/(\lambda^2 - 1))$. The formal Euclidean distance of the line (4.3) from the instanton is $[(\lambda^2 + 1)/(\lambda^2 - 1)]\sinh\mu$. The scale of the instanton divided by this distance is therefore $\sigma/\sinh\mu$, where σ is given by (3.6) with $c = 0$. Once again one recovers the same geometrical interpretation. Note that there is a particular section of $\mathbb{C}^4$ which is invariant under the transformation (4.2), namely $\{\boldsymbol{x} \in \mathbb{R}^3; x_4 = \mathrm{i}t, t \in \mathbb{R}\}$. The space $\mathbb{R}^{3,1} = \{(\boldsymbol{x}, t)\}$ is naturally Minkowskian: the lines (4.3) are the time-like lines through the origin, which may be identified naturally with the points of H^3.

Let us now turn to the interpretation of the extended ansatz, i.e. values of the scale, σ, greater than 1. We shall work with the moduli space of 1-instantons. It is convenient to factor out the constant gauge transformations and consider the five-dimensional space $M = M_1/\mathrm{SO}(3)$, whose coordinates $(\lambda \neq 0, \boldsymbol{c}, c_4)$ are the scale and the centre of the instanton. Locally, M may be regarded as $\mathbb{R}^5$, but the points with $\pm\lambda$ should be identified, since they correspond to the same instanton. The boundary $\lambda = 0$ may be identified with ordinary space, $\mathbb{R}^4$, and it is important to note that a conformal transformation T on $\mathbb{R}^4$ induces a conformal transformation $\tilde{T}$, on M, whose restriction to the hyperplane $\lambda = 0$ is T. Recall that the orbits of G acting on moduli space are the sets of instantons giving rise to the same Skyrme field. Just as the orbits of G in $\mathbb{R}^4$ are the circles (4.1), so the orbits of G in M are a similar family of circles C. In any plane containing the c_4 axis they form a coaxal system of non-intersecting type. For instance, in the plane $\boldsymbol{c} = \mathbf{0}$, (3.6) gives

$$
\left(\lambda - \frac{1}{\sigma}\right)^2 + c_4^2 = \frac{1}{\sigma^2} - 1\ . \tag{4.4}
$$

This should be compared with (4.1); the linear parameter is now $1/\sigma$. We see that there is a circle's worth of instantons which generates the Skyrmion of scale $\sigma < 1$. When

$\sigma = 1$ the circle degenerates to a point: the Skyrmion is generated by a G = SO(2)-invariant instanton. When $\sigma > 1$ the circle (4.4) has no real points. The corresponding gauge fields are still self-dual, apart from singularities, but the gauge group is now SL(2, $\mathbb{C}$) rather than SU(2).

By extending the coordinates of M to complex values we see there is no distinction between $\sigma \leq 1$ and $\sigma > 1$. In particular, consider the instanton with imaginary centre $(0, \mathrm{i})$ and scale λ. This generates a Skyrmion of size $\sigma = 2/\lambda$, which may be arbitrarily large. After the transformation (4.2), the instanton has the same centre, and new scale σ. Note that its distance from the line (4.3) is $\sinh\mu$, so the scale divided by this distance is again $\sigma/\sinh\mu$. More generally, we may consider the transformation induced by (4.2) on all the points of M. The circles C in M transform into lines through the origin. As before, we obtain an interesting representation by considering the restriction to the invariant section $\{(\lambda, c) \in \mathbb{R}^4; c_4 = \mathrm{i}\gamma, \gamma \in \mathbb{R}\}$. The points inside the light-cone $\gamma^2 = \lambda^2 + c^2$ represent the standard instantons; they generate the Skyrmions with the restricted scale. The points outside represent the gauge fields which give rise to the Skyrmions with the extended scale. On the light-cone itself, each light-like line corresponds to an SO(2)-invariant instanton i.e. a hyperbolic monopole.

We may summarise the results of these complex transformations as follows. To obtain a Skyrme field on S^3 (with $B = 0$) we evaluate the holonomy of an instanton of scale σ, centre $(0, 1)$ along the lines $x_4 = r/\tan\mu$. To obtain a Skyrme field on H^3 we evaluate the holonomy of an instanton of scale σ, centre $(0, \mathrm{i})$ along the lines $x_4 = \mathrm{i}r/\tanh\mu$. In each case σ and μ are real.

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